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DISCUSSION OF PROCEEDINGS - SEPARATES

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M. Z. v. KRZYWOBLOCKI[13].—An interesting approach has been presented by

[13] Prof. of Gas Dynamics and Theoretical Aerodynamics, Univ. of Illinois, Urbana, Ill.

the author to the problem encountered in the investigation of elastically supported plates. It appears that Mr. Fuchs has underestimated the method offered by Mr. Ritz.[6] Although this method does not usually bear Mr. Ritz's

[6] "Über eine neue Methode zur Lösung gewisser Variationsprobleme der mathematischen Physik," by W. Ritz, *Journal für reine und angewandte Mathematik*, Vol. 135, 1909, pp. 1–61.

name, the method has furnished many successful solutions in the theory of plates.

In 1908 Mr. Ritz proposed an approximation method for solving the problems of a variational character.[14] He proved that under some conditions

[14] "Variational Methods for the Solution of Problems of Equilibrium and Vibrations," by R. Courant, *Bulletin*, Am. Mathematical Soc., Vol. 49, 1943, pp. 1–23.

the minimizing sequence converges (but stated that the limit of convergence can be quite different from the sought for solution), and he applied his method to plate problems, vibrating-string problems, and the Dirichlet problem. Subsequent investigators have made use of Mr. Ritz's work but have failed to make the proper references to the initial investigation.[15,16,17] A comparison

[15] "The Approximate Solution of Two-Dimensional Problems in Elasticity," by S. Timoshenko, *Philosophical Magazine*, Vol. 47, Series 6, 1924, pp. 1095–1104.

[16] "Strength of Materials," by S. Timoshenko, D. Van Nostrand Co., Inc., New York, N. Y., 1930, Vol. 1, p. 326, Paragraph 7.1.

[17] "Theory of Elasticity," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1934, p. 150, Paragraph 43.

of Mr. Ritz's work with the subsequent investigations has been made by the writer.[18] It has been shown that the principle of least work (based on the

[18] "On the So-called Principle of Least Method," by M. Z. v. Krzywoblocki, *Proceedings*, First Midwestern Conference on Solid Mechanics, Univ. of Illinois, Urbana, Ill., April, 1953.

method proposed by Mr. Ritz), depending on the choice of the minimizing sequence, may yield a rapid convergence (as shown in some of the examples of S. Timoshenko) or nonsensical results (as shown by the writer). It can be assumed that each particular type of boundary-value problem is characterized by its own most proper minimizing sequence.

Other approaches (developed by S. Bergman) to the problem under investigation are the integral-operator method[19] and the method of solving the

[19] "Über Schubknickung von isotropen und anisotropen Platten," by S. Bergman, *Proceedings*, Third International Cong. of Applied Mechanics, Stockholm, Sweden, 1930, pp. 82–87.

boundary-value problem.[20] Mr. Fuchs did not mention this interesting

[20] "Neuere Probleme aus der Flugzeugstatik. Über die Knickung von rechteckigen Platten bei Schubbeanspuchung," by S. Bergman, *Prikladnaja Matematika i Mekhanika*, Vol. 2, 1935, pp. 207–224.

method, proposed by Mr. Bergman in 1930 and applied by the writer in 1948.[21]

[21] "A General Approximation Method in the Theory of Plates of Small Deflection," by M. Z. v. Krzywoblocki, *Quarterly of Applied Mathematics*, April, 1948, pp. 31–52.

Other methods which can be successfully applied are those that were developed by A. Weinstein[22,23,24,25] and R. V. Southwell.[26,27]

[22] "Étude des spectres des équations aux dérivées partielles de la théorie des plaques élastiques," by A. Weinstein, *Mémorial des sciences mathématiques*, No. 88, 1937.

[23] "On the Unified Theory of Eingenvalues of Plates and Membranes," by N. Arojszajn and A. Weinstein, *American Journal of Mathematics*, Vol. 64, 1942, pp. 623–643.

[24] "On the Bending of a Clamped Plate," by A. Weinstein and D. H. Rock, *Quarterly of Applied Mathematics*, Vol. 2, 1944, pp. 262–266.

[25] "On a Boundary Value Problem for a Clamped Plate," by A. Weinstein and J. A. Jenkins, *Transactions*, Royal Soc. of Canada, Section 3, 1946, pp. 59–67.

[26] "Relaxation Methods Applied to Engineering Problems," by L. Fox and R. V. Southwell, *Transactions*, Royal Soc. of London, Series A, Vol. 239, 1945, pp. 419–460.

[27] "Relaxation Methods Applied to Engineering Problems," by D. G. Christopherson, L. Fox, F. S. Shaw, and R. V. Southwell, *ibid.*, pp. 461–487.

R. H. Wood.[28]—An analytical solution which rarely has been attempted because of its complexity has been obtained by the author. The problem involved is the determination of the behavior of rectangular and square plates supported on elastic beams. It is not generally understood, however, what significance the solution has in relation to structural-engineering research.

[28] Prin. Scientific Officer, Building Research Station, Garston, Watford, Hertfordshire, England.

A great deal of attention has been given to the design of multi-story building frames both in the elastic field[29] and by "plastic" methods to determine the point of collapse.[30] The next logical step would be to determine what additional effects—in terms of both the behavior at working conditions and the enhancement of the collapse load—can be produced by the presence of cladding. When there are reinforced concrete floors in a building, it has been found that the reduction of stress in the supporting beams can be so large that the resulting frame action is considerably modified. In Fig. 10 there is shown a floor being subjected to a distributed load of 12 tons. The maximum recorded stress was approximately $\frac{1}{4}$ ton per sq in. The corresponding stress computed for bare frame action would be many times this amount. Consequently, the Building Research Station at Garston, Watford, Hertfordshire, England, has commenced a systematic study of "composite action" by analysis, controlled tests in the laboratory, and field tests. It will be shown that the problem which Mr. Fuchs has investigated is one of considerable practical importance.

[29] "An Economical Design of Rigid Steel Frames for Multi-Storey Buildings," by R. H. Wood, *Research Paper No. 10, National Building Studies*, Her Majesty's Stationery Office, London, England, 1951.

[30] "A Review of Recent Investigations into the Behavior of Steel Frames in the Plastic Range," by J. F. Baker, *Journal*, Inst. of C.E., January, 1949.

This particular floor-beam problem has been studied by the writer using the method of finite differences (for elastic behavior) and the "limit-analysis" method (for ultimate collapse), and a general report on the relationship of the problem to the study of composite action in multi-story buildings has been prepared.[31] In Fig. 11 there is shown the distribution of vertical reaction between floor and beam for a square slab with beams of equal stiffness. Fig. 11

[31] "Studies in Composite Construction, Part II," by R. H. Wood, *Research Paper No. 24. National Building Studies*, Her Majesty's Stationery Office, London, England (publication pending).

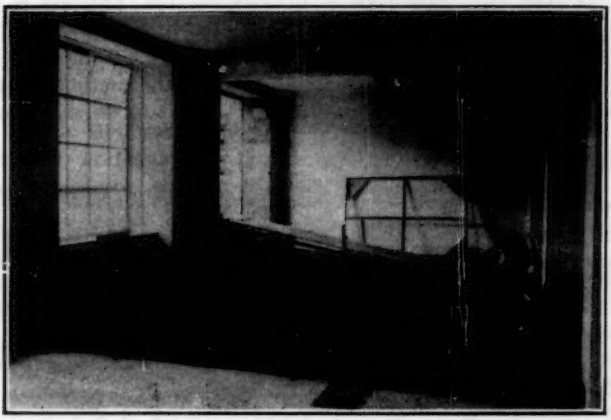


FIG. 10.—FLOOR UNDER LOAD

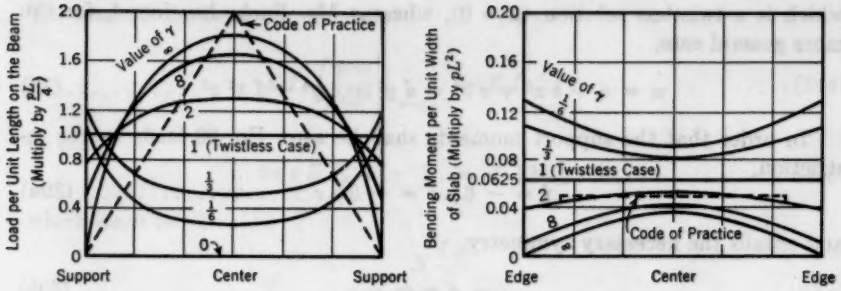


FIG. 11.—DISTRIBUTION OF LOAD INTENSITY ON THE SUPPORTING BEAMS ($\mu = 0$) FOR UNIFORM LOADING ON SQUARE SLAB

FIG. 12.—DISTRIBUTION OF BENDING MOMENT AT CENTER SECTION OF SLAB ($\mu = 0$) FOR UNIFORM LOADING ON SQUARE SLAB

has been drawn for the case in which Poisson's ratio is equal to zero. The British Code of Practice[32] allows the computations for elastic design to proceed on a basis of Poisson's ratio equal to zero, but as an alternative the beam loads, as shown by the dashed lines, can be used. It can be seen that Fig. 11 is of the same form as Fig. 7 after allowance has been made for the effect of Poisson's ratio. Similar agreement is found in the case of the bending moments across the center section (Fig. 12). The parameter relating beam stiffness ($E\,I$) to the flexural rigidity of the half slab $\left(N\frac{L}{2}\right)$ is

[32] "The Structural Use of Normal Reinforced Concrete in Buildings," *British Standard Code of Practice C.P. 114*, British Standards Inst., London, England, 1948.

$$\gamma = \frac{E\,I}{N\frac{L}{2}} \qquad (36)$$

It is apparent that, when γ is equal to unity, a "twistless" case ensues (when μ is equal to zero)—each strip acting independently and resulting in a uniform reaction on each beam of $\frac{p\,L}{4}$ per unit length. It is interesting to note that the author has found (Fig. 7) that a uniform load on the beams can be produced in certain cases even when Poisson's ratio is not zero—that is, when the condition is not that of zero twist. Of even greater interest is the fact that the writer has discovered that a uniform beam reaction can ensue at the point of incipient plastic collapse when there is a considerable amount of twist present. This phenomenon might be termed "reaction-redistribution" and is of considerable importance in determining the behavior of composite systems near collapse.

There are some interesting features in the polynomial expression for the deflection of a plate. The writer has investigated the deflection of a composite slab-beam system and has found that the deflection can be expressed as

$$w = a + b\,x^2 + c\,x^4 + d\,y^2 + e\,y^4 \qquad (37)$$

which is a twistless solution ($\mu = 0$), whereas Mr. Fuchs has found, for the more general case,

$$w = a + b\,x^2 + c\,x^4 + d\,y^2 + e\,y^4 + f\,x^2\,y^2 \ldots\ldots\ldots (38)$$

In order that the support moments shall be zero, Eq. 38 leads to the restriction,

$$f = -\,6\,\mu\,e = -\,6\,\mu\,c \ldots\ldots\ldots\ldots\ldots\ldots (39a)$$

and entails the necessary symmetry,

$$c = e \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39b)$$

which leads to the singular solution,

$$\gamma_x = \frac{E\,I_x}{N\,\frac{L_a}{2}} = \gamma_y = \frac{E\,I_y}{N\,\frac{L_b}{2}} = (1 - \mu)^2 \ldots\ldots\ldots\ldots (39c)$$

for a rectangle of sides L_a and L_b.

It is interesting to note, however, that, if μ is small enough to be neglected, there results a greater generality in the possible variations in beam stiffness. Commencing with Eq. 37, the bi-harmonic equation for the plate results in

$$c + e = \frac{p}{24\,N} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (40)$$

in which c is not equal to e. Also, since μ is equal to zero,

$$M_x = -\,N\,\frac{\partial^2 w}{\partial x^2} = -\,N\,(2\,b + 12\,c\,x^2) \ldots\ldots\ldots\ldots (41)$$

If M_x is equal to zero along the side $x = \pm\,L_a/2$,

$$b = -\,\frac{3}{2}\,c\,L^2_a \ldots\ldots\ldots\ldots\ldots\ldots\ldots (42a)$$

and by symmetry

$$d = -\,\frac{3}{2}\,e\,L^2_b \ldots\ldots\ldots\ldots\ldots\ldots\ldots (42b)$$

In the twistless case the beam-to-slab reaction is

$$V_x = Q_x - \frac{\partial M_{xy}}{\partial y} = -\,N\,\frac{\partial^3 w}{\partial x^3} = -\,24\,c\,x\,N \ldots\ldots\ldots (43a)$$

Along $x = L_a/2$, Eq. 43a can be written as

$$V_x = -\,12\,L_a\,c\,N \ldots\ldots\ldots\ldots\ldots\ldots\ldots (43b)$$

The load on the beam, however, is

$$E I_x \left(\frac{\partial^4 w}{\partial y^4} \right)_{x=L_c/2} = E I_x \, 24 \, e \quad \ldots\ldots (44a)$$

so that

$$24 \, e \, E I_x = - V_x = + 12 \, L_a \, c \, N \quad \ldots\ldots (44b)$$

which leads to

$$e = c \frac{N \frac{L_a}{2}}{E I_x} = \frac{c}{\gamma_x} \quad \ldots\ldots (45a)$$

and

$$c = \frac{e}{\gamma_y} \quad \ldots\ldots (45b)$$

For Eqs. 45 to be applicable simultaneously, it is only necessary to have the less restrictive condition,

$$\gamma_x = \frac{1}{\gamma_y} \quad \ldots\ldots (46a)$$

or

$$\gamma_x \gamma_y = 1 \quad \ldots\ldots (46b)$$

It will be noticed from Eqs. 46 that there is now no restriction on the magnitudes of the beam stiffnesses. However, in this twistless case if one set of beams is strengthened the other set must be weakened. The intensity of load on the beams is

$$\text{Along } x = \frac{L_a}{2} V_x = - \frac{p \, L_a}{2} \frac{1}{1 + \frac{1}{\gamma_x}} \quad \ldots\ldots (47a)$$

$$\text{Along } y = \frac{L_b}{2} V_y = - \frac{p \, L_b}{2} \frac{1}{1 + \frac{1}{\gamma_y}} \quad \ldots\ldots (47b)$$

Consequently, if Poisson's ratio is small, the reactions will be given approximately by Eqs. 47. However, because there is no restriction to the singular solution (Eq. 39c) it is easier to interpret the behavior of floor-beam systems.

This conception is enhanced when one attempts to determine if a twistless case can exist in continuous slab-beam systems, restricting the investigation to continuous rectangular panels supporting continuous uniform loading.

Assuming

$$N (1 - \mu) \frac{\partial^2 w}{\partial x \, \partial y} = 0 \quad \ldots\ldots (48)$$

at all points, then $\partial w / \partial y$ must be a function of y only. It follows that

$$\frac{\partial^4 w}{\partial y^4} = f''' (y) \quad \ldots\ldots (49)$$

Because this attempted solution represents, in effect, the behavior of independent strips, the load carried in the x-direction is not necessarily the same; therefore,

$$\frac{\partial^4 w}{\partial x^4} = \phi''' (x) \quad \ldots\ldots (50a)$$

from which

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \delta y^2} + \frac{\partial^4 w}{\partial y^4} = \phi''' (x) + 0 + f''' (y) = \frac{p}{N} \quad \ldots\ldots (50b)$$

If the load is shared in the proportion,

$$\phi''' (x) = \alpha \frac{p}{N} \quad \ldots\ldots (51a)$$

$$f''' (y) = (1 - \alpha) \frac{p}{N} \quad \ldots\ldots (51b)$$

then

$$\frac{\partial w}{\partial x} = \alpha \frac{p}{N} \frac{x^3}{6} + B\,x \quad \ldots\ldots (51c)$$

retaining only odd powers of x because of symmetry.

Since w is equal to zero at the corners,

$$w = \alpha \frac{p}{N} \left(\frac{x^4}{24} - \frac{L^2_a\, x^2}{48} + \frac{L^4_a}{384} \right) + (1 - \alpha) \frac{p}{N} \left(\frac{y^4}{24} - \frac{L^2_b\, y^2}{48} + \frac{L^4_b}{384} \right) \quad \ldots (52)$$

Along the edge, $y = L_b/2$, the combined reaction per unit length from the slabs on both sides is found to be $(1 - \alpha)\, p\, L_b$, and along the edge, $x = L_a/2$, the reaction is equal to $\alpha\, p\, L_a$ both of which are constant.

Because the beam is virtually built-in (because of continuity), the deflection of the long beams carrying this uniform load would be

$$w = \frac{1}{E\, I_y} (1 - \alpha)\, p\, L_b \left(\frac{x^4}{24} - \frac{L^2_a\, x^2}{48} + \frac{L^4_a}{384} \right) \quad \ldots\ldots (53)$$

but the slab deflection along $y = L_b/2$ is given by

$$w = \alpha \frac{p}{N} \left(\frac{x^4}{24} - \frac{L^2_a\, x^2}{48} + \frac{L^4_a}{384} \right) + \underbrace{(1 - \alpha) \frac{p}{N} \left(\frac{L^4_b}{384} - \frac{L^4_b}{192} + \frac{L^4_b}{384} \right)}_{\text{zero}} \quad \ldots (54)$$

Eqs. 53 and 54 are identical if

$$\frac{1}{E\, I_y} (1 - \alpha)\, p\, L_b = \alpha \frac{p}{N} \quad \ldots\ldots (55a)$$

or

$$\frac{E\, I_y}{N\, L_b} = \frac{1 - \alpha}{\alpha} = \gamma_{c,y} \quad \ldots\ldots (55b)$$

in which the subscript c denotes continuity, and each beam is now associated with a total width of slab of L_b.

Similarly,

$$\frac{E I_x}{N L} = \frac{\alpha}{1-\alpha} = \gamma_{c,x} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56a)$$

from which

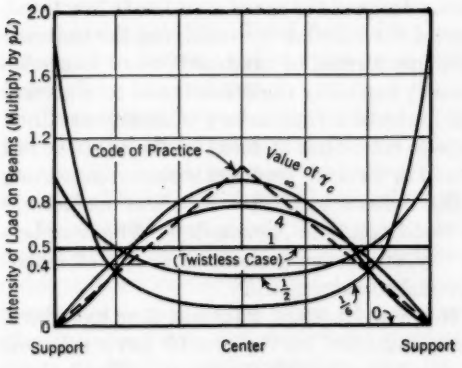
$$\gamma_{c,x}\, \gamma_{c,y} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56b)$$

which is the condition relating slab and beam stiffnesses if the twistless case is to occur. This condition is valid for all values of Poisson's ratio, whereas the condition was only valid for μ equal to zero in the case of isolated panels. Of greater interest is the fact that Eq. 56*b* is exactly the same as the "less restricted" condition given by Eq. 46*b*. These twistless cases therefore serve as a landmark in a general study of the subject.

Fig. 13 shows the results obtained for the beam loads which appertain to continuous, square panels. Fig. 13 emphasizes the importance of studying the load distribution on the beams. The finite-difference results contain small errors which could be improved by using a finer mesh in the computations. The twistless case shows up prominently.

A brief introduction to the collapse of such slab-beam systems is of interest. Fig. 14 shows the collapse of a symmetrical square panel. The collapse mode is identified by the fracture lines (lines of full plastic moment) in the slab approximately along each center line and by the plastic hinges in the beams to form a mechanism. Above a certain critical ratio of beam-to-slab carrying capacity, theory indicates that the rectangular mode of collapse is abandoned in favor of a diagonal collapse of the slab. The supporting frame does not take part in this collapse. There is clearly a limiting load distribution collected by the beams, which is eventually insufficient to cause the beams to collapse. That is, this limiting load distribution is decided by the slab and beams working in unison. In general, it cannot be decided intuitively in advance and then coupled with a load factor.

Design on such a basis may, however, be intuitively correct in special cases.[31] Consequently, Mr. Fuchs' analytical solution is a welcome and valuable addition to the information which is required in the study of composite structures in general.



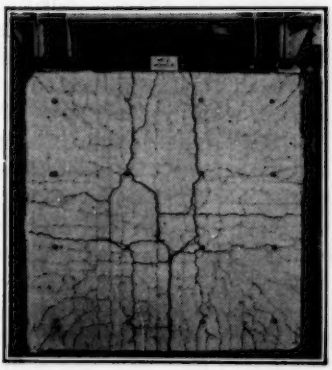


FIG. 13.—APPROXIMATE DISTRIBUTION OF LOAD INTENSITY ON THE BEAMS SUPPORTING CONTINUOUS SQUARE SLABS SUBJECTED TO CONTINUOUS UNIFORM LOADING

FIG. 14.—UNDERSIDE OF A COLLAPSED SLAB-BEAM SYSTEM

S. J. FUCHS,[33] M. ASCE.—Interesting comments concerning developments in various aspects of the investigation of elastically supported plates have been offered by Messrs. Krzywoblocki and Wood.

[33] Chf. Stress Analyst, Loewy Construction Co., New York, N. Y.

Mr. Krzywoblocki's justifiable criticism is that the writer's brief comment relative to the method offered by Mr. Ritz appears to underestimate the method. Under the heading, "Introduction: Historical Note," it was stated that the usual application of the Rayleigh-Ritz method involves a minimizing sequence

$$w_n = c_1\psi_1 + c_2\psi_2 + \cdots + c_n\psi_n \ldots\ldots\ldots\ldots\ldots\ldots (57)$$

in which each coordinate function ψ_i satisfies the boundary conditions. However, functions are not readily available which satisfy the boundary conditions of the typical plate, simply and flexurally supported, which has been analyzed in detail. This problem can be investigated by use of the strain-energy method with dependence on natural boundary conditions; that is, the integral expression for strain energy D_n (w_n, λ, κ)—with dependence on the parameters λ and κ—can be minimized with respect to the coefficients c_i. This minimum evaluation of D_n may not be affected intimately by the choice of coordinate functions ψ_i, and similarly the evaluation of a scalar such as a characteristic value (also derived by integration) may likewise be unaffected directly. The evaluation of other properties, however, such as shear (which depends on third-order derivatives) is comparatively directly affected by the choice of functions ψ_i. By the formulation of the problem without the aid of functions which satisfy the conditions as imposed, the error involved in the evaluation of a property is probably increased. Furthermore, the strain-energy method as applied to the case of the typical, flexurally supported plate requires the solution of simultaneous linear equations—as is also necessary in the method offered by the writer.

In 1953 a strain-energy analysis of the clamped, flexurally supported plate was reported[34] by J. G. Sutherland, J. M. ASCE, L. E. Goodman, and N. M. Newmark, M. ASCE. Their problem admitted the use of coordinate functions, referred to previously, which facilitated the solution by satisfying the imposed boundary conditions. Certain functions, formed of combinations of Legendre polynomials, were utilized so as to satisfy explicitly the condition of zero normal slope around the boundary. For the sake of a comparison of results obtained by the alternative methods, some typical maximum moment coefficients derived by the strain-energy method[34] are listed in Table 3, together with corresponding coefficients obtained by the use of the writer's method. No shear coefficients were listed by Messrs. Sutherland, Goodman, and Newmark.[34] The complete analysis and numerical evaluation (by the writer's method) of the clamped, flexurally supported plate may be published subsequently.

[34] "Analysis of Plates Continuous over Flexible Beams," by J. G. Sutherland, L. E. Goodman, and N. M. Newmark, *Technical Report, Project NR-064-183*, Office of Naval Research, Washington, D. C., January, 1953.

The historical note was limited to a few standard solutions that have been applied to plate problems. A more extended survey would have required mention of the important methods developed by Mr. Bergman. The latter's

integral-operator method, alluded to by Mr. Krzywoblocki, provides a solution for an assigned, linear, partial, differential equation as a transform of arbitrary functions. The method, however, appears to offer no particular facility in dealing with the boundary conditions typical of this subject. Futhermore, the method requires involved computations.[19,20] The writer's method is considered to be sufficiently general for many applications in engineering and was shown to reduce to standard solutions for particular cases.

Mr. Wood's informative account of research on composite panel action conducted at the Building Research Station at Garston (England) indicates the alertness of investigators to the implications, in practice, of developments in the theory of transversely loaded panels. The coefficients for moment

TABLE 3.—MAXIMUM-MOMENT COEFFICIENTS FOR CLAMPED PLATES[a]

Coefficient	VALUE[b] OF λ_2/a AND λ/b				
	0.5	0.25	0.5	0.25	0.1
By S. J. Fuchs' method	VALUE OF b/a				
	1.0		0.75		
$-C_{mx}\vert_{x=a/2,y=b/2}$	1/24	0.069043	0.035715	0.057866	0.097062
$-C_{my}\vert_{x=a/2,y=b/2}$	1/24	0.069043	0.026785	0.046045	0.082592
By strain-energy method[c]	VALUE OF b/a				
	1.0		0.80		
$-C_{mx}\vert_{x=a/2,y=b/2}$	1/24	0.06904	0.03704	0.06010	0.09951
$-C_{my}\vert_{x=a/2,y=b/2}$	1/24	0.06904	0.02963	0.05037	0.08775

[a] At the corners of flexurally supported clamped rectangular plates ($\mu=0$) under uniform load, p. Thus, $M_x = C_{mx}\, p\, a^2$, and $M_y = C_{my}\, p\, a^2$. [b] Values of parameter λ refer to single panels, or one half of the total ridigity of the beam between adjacent panels. [c] As reported by Messrs. Sutherland, Goodman, and Newmark.

and shear which Mr. Wood derived by another alternative method—that of difference equations—are of great value in providing the mutual corroboration required by the engineer.

The exact solution for the clamped plate classification can be derived conveniently by recognizing that the normal slope, hence the mixed derivatives w_{xy}, vanish at each point on the boundary. Thus, it is sufficient to set f equal to zero in Eq. 38, and Eqs. 43 to 46 follow as shown.

It is agreed that elastic stress, computed (for example) by the coefficients from Table 3, are also of value in consideration of plastic stress. Thus, the residual stress caused by plastic action can be computed as the difference between plastic and elastic stress at a generic point provided that the residual effective shear does not exceed the yield strength in unloading. Thereafter, cycles of loading and unloading may be all-elastic; in these cases, elastic coefficients are necessary.

Flexurally supported panels are important in structural design. It has been usual to consider panels to be supported on beams of implied complete rigidity ($\lambda \rightarrow \infty$). The analyses and numerical data which have been made available for flexurally supported plates give the designer greater latitude and enable him to use supporting beams of arbitrary rigidity.

DISCUSSION OF APPLICATION OF THE RELAXATION TECHNIQUE IN FLUID MECHANICS
PROCEEDINGS—SEPARATE 223

JOHN S. McNOWN[1], M. ASCE, EN-YUN HSU[2], A.M. ASCE, and CHIA-SHUN YIH[3].—In the several discussions submitted, contributions have been made which range from considerations of the philosophy underlying numerical methods through matters of technique to comparisons of numerical and observed results. The fact that only two discussers described the conduct of additional calculations is probably an indication that the method has not yet found general use. If the paper and the discussions serve to stimulate such use, their primary objective will have been achieved.

Mr. Blaisdell considered primarily the accuracy, or rather the consistency of the numerical results for flow over a weir. The obvious lack of system at one point in his table led to the finding of a typographical error in the published results. For the upper nappe, the value of y/H for $x/H = \frac{7}{8}$ should have been 0.442 instead of 0.422. Aside from this erroneous value the results of his equation and of the relaxation are remarkably close together.

Although Mr. Blaisdell characterized a part of the correspondence as erratic, Messrs. Paynter and Scott apparently found the general agreement between the results observed at the Bureau of Reclamation and those of the relaxation almost too good to believe. Actually, these data came to the writers' attention well after both the calculation and the manuscript had been completed. Had they been used in the calculation, reference would, of course, have been made to them. Mr. Frank B. Campbell of the Waterways Experiment Station informed the writers of their existence in a letter, and Mr. J. N. Bradley of the Bureau of Reclamation submitted a comparison like that of Messrs. Paynter and Scott. Unfortunately, his comments were received too late to be published as a discussion.

A comparison between second differences can be misleading, particularly if one set is calculated from a quadratic equation for which the second differences are inherently constant. As their order of magnitude in this calculation is approximately $H/100$, presentation of the second differences as 3- and 4-digit numbers by Blaisdell is misleading. Even for his empirical equation, the rounding-off process results in a sizable change.

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The trend of the values for the thickness of the jet is more significant. However, the value of 0.5 for x/H is not large enough for the internal pressure to be ignored, as is clearly indicated in Fig. 13 of the original paper. For $x/H \approx 1$, the internal pressure head is still more than $H/10$. Thus the conditions of free fall assumed by Blaisdell are only approximately valid. The fact that the observed thickness differs from the mean of the calculated thicknesses by less than one percent seems quite remarkable.

Messrs. Paynter and Scott objected to the absences of both a broader statement of the underlying philosophy of numerical methods and fuller numerical detail. In their comment, "Rather than merely being techniques of numerical integration, . . . " the word "merely" is the discussers' and not the writers'. Their more general interpretation of the method is in accord with the writers' views. It had already been amply stated in such references as those supplied by the discussers. Omission of the relative width ($b = 2h/3$) of the permeable wall in Fig. 17 was an oversight.

Numerical details were included here and there in the paper more as illustrations of what can be done than as exhaustive solutions to practical problems. In this connection the computations for weir flow (Fig. 13) and for seepage (Fig. 17) are hardly comparable in significance. The former is a classic problem of considerable practical importance; the latter is an isolated example which is only illustrative of an infinite variety of boundary configurations. Furthermore, difficulties inherent in the presentation of the numerical results will be obvious to the reader. In the cavity flow (Fig. 14), for example, over 1300 values of the stream function were determined. To have gone very far in either of the somewhat opposing directions proposed by Paynter and Scott would have unduly extended an already long paper.

Boreli and Paynter and Scott described methods and conditions for which a mesh other than square is used. Some of these were also discussed in Southwell's original work on the subject. Aside from the square mesh, however, only the rectangular which Boreli used to advantage in calculating seepage toward wells is likely to be applicable to problems of fluid flow. The rather distantly related Monte Carlo method is interesting and perhaps useful for isolated applications.

Mr. Boreli's contention that the formula obtained by Taylor's expansion for the residue is often preferable to the one presented in the paper is readily conceded. In fact, in using the relaxation technique to determine the cavity shape (Fig. 14), one of the writers used the formula from Taylor's expansion. It must be noted, however, that if the second and higher derivatives of the unknown function are large, and the net is comparatively coarse, the superiority of Eq. (2) to Eq. (1) in Mr. Boreli's discussion cannot be established, since Eq. (2) is then only a rough approximation. All one can say is that as the net becomes finer, the accuracy of Eq. (2) is increased more rapidly than that of Eq. (1).

Mr. Boreli proposed a treatment of singular points by properly weighting the near-by legs. Unfortunately, he did not mention how the weighting is to be determined. The treatment presented by Mr. Sarpkaya for the same purpose is simple and systematic and is a welcome contribution to this troublesome aspect of the relaxation technique.

Although the writers stated that application of the relaxation method is feasible only for two-dimensional or axisymmetric flows, recent studies[4,5] have included analyses of problems for which variations in all three dimensions are significant. Mr. Roy obtained a rough approximation for flow through a transition from a reservoir to a rectangular conduit, and Messrs. Allen and Dennis demonstrated a useful technique which is particularly well suited to problems for which the third dimension is small compared to the others. A solution of acceptable accuracy for a three-dimensional flow with a free surface, as an extreme example, remains an exorbitant undertaking.

The discussers indicated that they share the writers' convictions as to the applicability and versatility of the relaxation technique. Mr. Sarpkaya's application of the method to problems of wave motion is additional evidence of its adaptability to a wide variety of potential flows. By their contributions, the discussers have expanded and reinforced various phases of the study.

4. Allen, D. N. de G. and Dennis S. C. R., "The application of relaxation methods to the solution of differential equations in three dimensions". I. Boundary value problems. Quart. Sr. of Mech. and Appl. Math., Vol. IV Part 2, June 1951, pp 199-208.

5. Roy, S. K., "Development of three dimensional relaxation processes for a systematic approach to flow analysis in hydraulic structures" (In three parts), Irrigation and Power (Indian Jr.) Vol. 9, No. 1, Jan. 1952.

DISCUSSION OF NUMERICAL ANALYSIS OF FRAMES WITH CURVED GIRDERS PROCEEDINGS—SEPARATE 250

JAMES MICHALOS[1], M. ASCE.—Mr. Conwell's discussion has added to the value of the paper, and his very kind comments are greatly appreciated.

An interchange of subscripts within the parentheses in the notation for carry-over may be desirable. The arrangement chosen seemed convenient, but this is largely a matter of preference. However, because this paper is related to another[2], and on occasion use of both may be necessary for an analysis, some additional remarks regarding notation are pertinent. In the latter paper, written and presented previous to the paper now under discussion, a prime mark is used to indicate carry-over to the opposite end of a member. In the paper now under discussion it was necessary to use a more specific designation, such as A-B and B-A, for carry-over factors.

Tables or curves of properties are not at present available for other than members of circular-arc plan and constant cross section but, as stated in the paper, values of stiffnesses, carry-over factors and fixed-end moments can be determined for any shape and variation in cross section by application of the shear and torsion analogy presented elsewhere[3]. In that paper it is shown that moments resulting from load applied normal to the plane of a structure fixed at its ends can be computed directly by a numerical procedure in a manner analogous to the computation of stresses due to direct shear and torsion in a strut. The procedure is illustrated in Figs. 6 and 7 of that paper. The reader should observe the difference in sign convention used in that paper and this one, and should also observe that the y-axis of that paper is the z-axis of this one. With these differences accounted for, the fixed-end moments in Fig. 7 of the previously published paper become $m_x = -8.79$ and $m_z = -15.01$ at A, and $m_x = -2.46$ and $m_z = +4.81$ at B.

When account is taken of the differences in axes and signs, the stiffness at an end A of a member AB, where end A is the right-hand end, and the corresponding carry-over factors (in terms of the stiffness) can be computed as follows:

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2. "Numerical Analysis of Continuous Frames in Space", by James Michalos, Proceedings—Separate No. 261, ASCE, September, 1953.

3. "Laterally Loaded Plane Structures and Structures Curved in Space", by Frank Baron and James Michalos, Transactions, ASCE, Vol. 117, 1952, p. 279.

$$K_{zA} = \frac{1}{A'_z} + \frac{(x_c)\,\mathring{A}}{J_y}$$

$$K_{zA}(C_{xz})_{A\text{-}A} = -\frac{P'_x}{A'_x} + \frac{(x_c)_A\,(z_c)_A}{J_y} \qquad (2)$$

$$K_{zA}(C_{xz})_{A\text{-}B} = \frac{P'_x}{A'_x} - \frac{(x_c)_A\,(z_c)_B}{J_y}$$

$$K_{zA}(C_{zz})_{A\text{-}B} = -\frac{1}{A'_z} - \frac{(x_c)_A\,(x_c)_B}{J_y}$$

To obtain K_{zB} and the corresponding carry-over factors, interchange all A and B subscripts in the preceding expressions. To obtain K_{xA} and the corresponding carry-over factors substitute x for z and z for x in the expressions. To obtain K_{xB} interchange the A and B subscripts as well as z and x.

If symmetry exists, as in Fig. 7 of the paper[3] to which reference was made previously, or if all elements of the member are parallel to one or the other of the reference axes, as in Fig. 6 of the same paper, then both P'_x and P'_z are equal to zero, and A'_x and A'_z are equal, respectively, to A_x and A_z. As an example of the use of Eqs. 1, a stiffness and the corresponding carry-over factors (in terms of stiffness) will now be obtained for the parabolic arch of variable section shown in Fig. 7. First, however, it is necessary that ends A and B of Fig. 7 be interchanged to conform to the nomenclature used in the paper now under discussion. Then, for end B (end A in Fig. 7),

$$K_{zB} = \frac{1}{84.38} + \frac{(-50)^2}{62{,}086} = 0.0521\ EI$$

$$K_{zB}(C_{xz})_{B\text{-}B} = -\frac{0}{70.04} + \frac{(-50)(-9.42)}{62{,}086} = 0.0076\ EI$$

$$K_{zB}(C_{xz})_{B\text{-}A} = \frac{0}{70.04} - \frac{(-50)(-9.42)}{62{,}086} = -0.0076\ EI$$

$$K_{zB}(C_{zz})_{B\text{-}A} = -\frac{1}{84.38} - \frac{(-50)(50)}{62{,}086} = 0.0285\ EI$$

where I is the moment of inertia about the z-axis at the crown.

Tables of fixed-end moments, stiffnesses and carry-over factors for any shape can be prepared by writing general algebraic expressions for the various terms of the shear and torsion analogy. The tables for members of circular arc plan were prepared in that manner.

Mr. Conwell outlines a straight-forward procedure for obtaining fixed-end moments due to a uniform load applied over a portion of a circular girder. Such values can also be obtained directly, as implied by Mr. Conwell, from influence lines. Influence lines for fixed-end bending moments are shown in Fig. 5. These were drawn by plotting values from Table 3. Influence lines for fixed-end twisting moments can be drawn in the same manner. To illustrate the method of obtaining

fixed-end moments due to partial loading, assume a circular girder AB whose central angle, ϕ, is equal to 90 degrees. Call its arc length L. Furthermore, consider this girder subjected to a uniformly distributed load, w per unit length of arc, over two-thirds of its length starting at end A. The fixed-end bending moment at A is equal to $\frac{wrL}{}$ times the area under the curve in Fig. 5 marked $\phi = 90$, between values of ϕ_a/ϕ equal to zero and 0.667. The fixed-end bending moment at B is equal to $\frac{wrL}{}$ times the area between the values of ϕ_a/ϕ equal to 0.333 and 1.000. Fixed-end twisting moments can then be found from similar curves, and those values are combined with the fixed-end bending moments, as shown in Eqs. 1, to obtain fixed-end moments about x- and z- axes.

The writer believes that the use of continuous girders curved in plan and the use of continuous frames with curved girders will increase. This will be true for bridges and elevated expressways, as well as for buildings and foundation frameworks, as witness the new Highbridge Interchange at the George Washington Bridge in New York[4]. The procedure presented in this paper readily yields a solution for the most complicated cases, solution of which is virtually impossible by other available methods.

Correction for Transactions.—In Fig. 5, the horizontal scale should be labeled "Values of ϕ_a/ϕ".

4. "Box Girder on Stilts", by Howard E. Reid and Phillips H. Lovering, Engineering News-Record, Feb. 7, 1952, p. 28.

DISCUSSION OF MECHANICS OF MANIFOLD FLOW

PROCEEDINGS—SEPARATE 258

JOHN S. McNOWN[1], M. ASCE.—Most significant among the numerous contributions of the discussers is the convincing evidence of the broad interest and wide range of applicability for studies of manifold flow. Applications specifically illustrated were the rapid sand filter mentioned by Mr. Barton, the sprinkler irrigation system described in useful detail by Messrs. Carstens and Roberson, and the lock manifold systems discussed by Mr. Hartigan. Interest is also demonstrated by the number of experimenters who have studied the problem. To be added to the list referred to in the paper are the Japanese experiments mentioned by Mr. Powell, and the Stanford work described by Messrs. Vennard and Dentoni.

One of the most troublesome problems in any experiment is the assessment of the probable accuracy of the data. Mr. Barton's fuller explanation of the procedures followed and the nature of the results is therefore welcome. In particular, he clearly expressed the importance of determining independently for the various pipe sections the piezometric gradient as a function of the discharge. This measure undoubtedly reduced both the scatter of the data and the absolute error of the trends established.

Additional results and alternative methods of presenting the data were contributed by Messrs. Escobar and Niaz, who also had participated in the experiments on which the paper was based. Mr. Escobar demonstrated convincingly the way in which the transverse velocity distribution causes the apparently negative head losses of Fig. 4, and observed the profile of the contracted jet in the lateral. In Fig. 4 of his discussion he illustrated the good correspondence between experiment, empiricism, and theory. Mr. Niaz presented in his discussion alternative plots, Figs. A and B, for the combining flow. These show (1) the degree of similarity of the loss to that in an abrupt contraction, and (2) the efficacy of plotting the head loss for the lateral flow against the ratio of the final to the initial velocity for this flow. Although the latter plot introduces a degree of system to the results, the data in Fig. B appear on close inspection to be three curves, each involving a transition from the lower envelope curve to the upper one. The results for this type of flow are evidently difficult to depict in any completely satisfactory manner.

The question of the minimum length of conduit or lateral to establish the intended uniform flow was raised in two of the discussions. The velocity distribution approaches its ultimate form asymptotically with

1. Prof., Univ. of Michigan, Dept. of Eng. Mechanics, Ann Arbor, Mich.; formerly Associate Director, Iowa Inst. of Hydr. Research, State Univ. of Iowa, Iowa City, Iowa.

distance downstream from the point of separation or major disturbance. Selection of the distance required for the branch point no longer to have an effect depends very much upon the purpose for which the determination is to be made.

The results shown by Mr. Hartigan are obviously, as he states, subject to action related as much to the jet velocity as to the mean velocity for the section, the lateral lengths involved being 2 or 3 lateral diameters. In a distance of 5 to 10 diameters the primary characteristics of the separation and high-velocity jet disappear. The relevant questions asked by Messrs. Carstens and Roberson can be given only a provisional answer. For a sprinkler unit to operate in a normal manner, again a distance downstream of only a few diameters would be necessary. Aside from the indication of the major flow conditions, the proposal to use the pressure taps as an indication of the degree to which the flow has been established is not reliable. The performance of a subsequent lateral, which was tested thoroughly by Yanes (4), was surprisingly dependent upon small changes in the velocity distribution. These changes cannot be detected accurately without a large number of carefully made piezometers, and, even then, the significance of a small change in pressure would probably have been underestimated without the direct measurements of lateral performance.

Detection by Messrs. Vennard and Dentoni of the error both in labelling and in discussing the comparison of the Munich and Iowa experiments in Fig. 4 is greatly appreciated. Although most of the results correspond quite well, those for this case certainly do not. The differences are not large, but the trends are opposite. The comparable data from Stanford, generally confirming the Iowa test results, are thus particularly valuable; the summary plots presented by Messrs. Vennard and Dentoni should also be useful. There are several minor questions about these, particularly Fig. 4. The curves should logically approach the limiting value of unity as Q_l/Q approaches zero. The curve for $D_l/D = 1.0$ is too high at the other end of the scale ($Q_l/Q = 1.0$). The suggested value of 1.5 is considerably greater than either the Munich value 1.3 or the Iowa value of 1.15. No data were given for $D_l/D = 1$ in the Stanford tests. If this curve were lowered as it apparently should be, the indicated trend of the curves would be improved. Omission of numerical values at either end of the scales for the various figures is needlessly confusing to the reader, particularly for the logarithmic scales.

Also welcome is the corroboration provided by the additional data of the Japanese experimenters for combining flow, referred to in Mr. Powell's discussion. As Mr. Powell pointed out, the correspondence among the several sets of data is quite good. For $D_l/D = 0.5$, the Iowa and Munich data differ somewhat (Figs. 7 and 8), and in these instances the data from Japan are very close to those from Iowa.

No matter how much care is given to the averaging of the various sets of data, the result will not be a final answer to all problems of manifold flow. In the paper it was emphasized that the data presented are for a single geometrical form of branch point or junction, and that the results obtained are representative rather than general. The variety of results presented by Mr. Hartigan for various lock ports well

illustrates the variability with port geometry of one characteristic of the flow. For this reason, extreme refinement either in the processing of the data or in its application to any such design problem as the sprinkler system is unjustified.

Various writers attach importance to different phases of symbology. Although the word friction is a misnomer for describing any kind of head loss in fluid flow, the symbol h_f for head loss is well known, and a lower-case subscript is more suitable to the typewriter than is a capital one. The terminology involving multiple subscripts, which Messrs. Vennard and Dentoni propose, seems needlessly complex. The system employed in the paper has the advantages of simplicity and of a maximum amount of carry-over between the converging and diverging types of flow.

By way of conclusion, the writer is gratified by the interest shown in discussions equivalent in length to nearly twice that of the paper itself, and by the fact that three of those who participated in the Iowa tests contributed additional findings from their thesis investigations. The goals of the Society in fostering thorough discussion were met admirably by the discussers in that varied points of view were expressed, additional results were presented, and various aspects of the findings were given special emphasis. The representative results obtained, reinforced in the discussions, should find application to a wide variety of practical problems.

DISCUSSION OF INELASTIC BEHAVIOR OF REINFORCED CONCRETE MEMBERS SUBJECTED TO SHORT TIME STATIC LOADS
PROCEEDINGS—SEPARATE 286

L. H. N. LEE[1], A. M. ASCE.—The author wishes to thank the discussers for their interests in the paper. Mr. Bresler's view that the precise shape of the flexural stress-strain curve of concrete is primarily of academic interest is correct as the variations of properties of concrete are large. However, there is no doubt that a certain stress-strain relationship, which can be mathematically expressed and manipulated, representing the average properties of concrete is needed. For studying the behavior of statically indeterminate structures, buckling of columns and other problems, the strength-deformation relationships are required; hence, in turn, the stress-strain relationship should be established.

It is confirmed that the stress-strain curve for concrete tested in standard compression cylinder test is a parabolic curve. It is known that the state of stress at the compression side of a beam is identical with the state of stress at a point inside a compression cylinder. The stress-strain relationships of two elements made of identical material and subjected to the same state of stress for the same duration obviously can not be different. It has also been early observed by A. N. Talbot as he stated in 1906:[16] ". . . . Various curves have been proposed to represent the stress-deformation relation for concrete, but the parabola is the most satisfactory general representation. Frequently, the parabola expresses the relation almost exactly and in nearly every case the parabolic relation will fit the stress-deformation diagrams very closely through the part which is ordinarily developed in beams. . . ."

The author has compared the deflections of eccentrically loaded columns tested by Hognestad,[21] with the theoretical deflections computed by the Equations (15) to (29). A total of 95 square and cylindrical columns were used in the comparison. The theoretical and actual deflections agree reasonably well as shown in Fig. 16.

The reinforcing bars used in the author's experiments were made of mild steel having a sharp yield point. This type of steel is commonly used in construction. The stress-strain curve of the steel is almost exactly represented by the idealized stress-strain curve shown in Fig. 4. In this case, the proportional limit and the yield point are very close to each other.

Mr. Bresler's example of using Equations (3) and (4) is not correct. Equation (3) is for the range of $0<\epsilon_s<\epsilon_y$ and Equation (4) is for the range of $\epsilon_y<\epsilon_s<\epsilon_{wh}$. The point $\epsilon_s=\epsilon_y$ is a point of discontinuity in

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16. Refer to footnote on page 286-10.
21. Refer to footnote on page 286-13.

first derivative. The values of $d\epsilon_s/d\epsilon_c$ for the left side and for the right side of the yield point are not the same.

Equation (3') suggested by Mr. Bresler is useful for the range when steel stresses are between proportional limit and yield point.

It is correct that calculation of flexural stress-strain relation involves numerical differentiation which may be very sensitive to the accuracy of experimental measurements as pointed out by Mr. Bresler. It has been experienced by the author that tests should be carried out with uniform increment of load in uniform time interval. Total time of testing should be short and special attention is needed at the ultimate load stage. It has also been the author's experience that use of a simply-supported beam will not give a stress-strain relation over a large range of strain, because the rapid collapse of the beam after it has absorbed a maximum of strain energy prevents the taking of an adequate number of accurate readings.

It should be pointed out that the test results with the strains less than 6×10^{-4} shown in Fig. 15 as presented by Mr. Bresler are not valid for the assumption of no tensile stress in concrete is not valid in that range.

The method of obtaining flexural stress-strain relationship suggested by Mr. Ashdown is questionable; because under the suggested condition, there is no way to preserve the condition of plane remains plane which is a basic assumption in forming the stress-strain relationship.

The readings shown in Table I stopped at 98% of the yielding moment of the section and 84½% of the collapse load of the restrained beam as clearly shown in Fig. 13. The strains beyond that load were not obtained, because the strains changed swiftly at that stage. The distribution of stresses as estimated by Mr. Ashdown is actually shown in Fig. 6.

The author does not agree with Mr. Ashdown that the ordinary engineer will be frightened by the complicated formulae given in the paper. However, if Mr. Ashdown's estimate of the "Ordinary Engineer" is valid, there remains the possibility of putting the results in tabular and graphical form, convenient for computation.

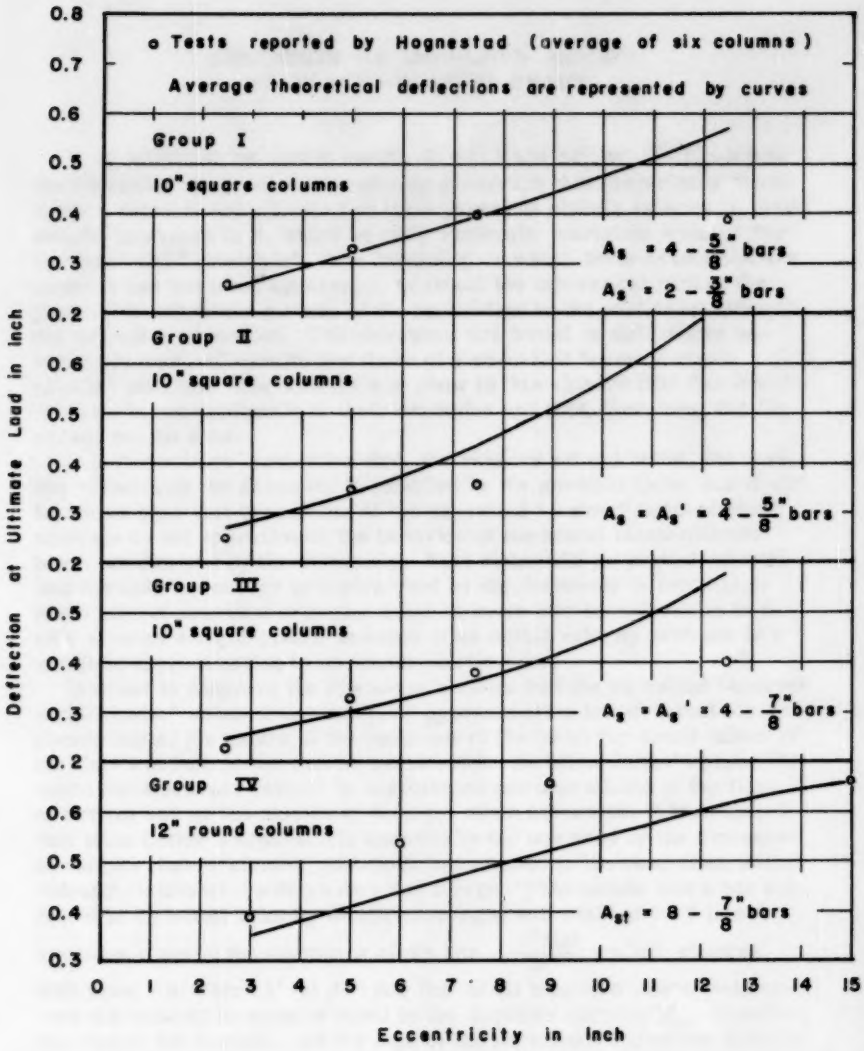

Fig. 16 Theoretical Deflections of Eccentrically Loaded Columns Compared with Test Results

DISCUSSION OF IMPULSIVE MOTION OF ELASTO-PLASTIC BEAMS

H. H. BLEICH[1], M. ASCE and M. G. SALVADORI[1], M. ASCE.—While the discusser concedes in his opening paragraph that the writers "have made a valuable contribution" in their paper, he objects later on to the results contained in it, which he calls variously "certainly wrong," "erroneous" and "untenable". The reasoning on which these objections are based is not intended, apparently, to attack the theoretical part of the paper, but rather the manner of its application to the particular numerical examples presented. The objections are based on differences between the writers' results and those of a so-called "correct rigid-plastic" analysis. The writers will show in this closure that this latter analysis is not applicable to their examples and that, therefore, the discusser has no case.

It is the writers' contention that, for reasons set out below, the manner of handling the examples is justified by the physical facts, and it will be shown here that the results of the so-called "correct rigid-plastic" analysis do not approximate the behavior of the actual elasto-plastic beam, as claimed by the discusser. This claim will be proved unjustified because the energy criterion used by the discusser is incomplete and a second essential criterion must be taken into consideration to decide whether a rigid-plastic solution of an initial velocity problem is a realistic approximation to an elasto-plastic case.

In order to disprove the discusser's claim that the so-called "correct rigid-plastic" solution is a realistic approximation for an actual elasto-plastic beam, the nature of the response of the beam for small values of the time t in both cases will be considered. As Miss Cotter's paper (to which the discusser refers) is unpublished and unavailable at the time of writing and as the discusser does not state otherwise, it is assumed that Miss Cotter's approach is essentially the one used by the discusser in various papers already published, but adapted to the case of an initial velocity distribution without external forces. This means that a bar subjected to an initial velocity distribution $y_0(x)$ will yield at $t = 0$ in all locations x where the curvature of the bar $\frac{d^2y(x)}{dx^2} = y''(x)$ changes with time, i.e. where $\dot{y}''(x) \neq 0$; and that in all locations where yield occurs the bending moment is equal to the capacity moment M_0. However, the sign of the moment, and the sign of the permanent curvature developing in any location depend on the sign of $\dot{y}(x)$. After the assumed inception of yielding at $t = 0$, but for very small values of t the deformation $y(x, t)$ is proportional to $\dot{y}_0(x)$, or

$$y(x, t) = t\,\dot{y}_0(x) \qquad \text{[for very small t]} \qquad (1)$$

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and the time rate of change of the curvature, the sign of which controls the direction of the permanent curvature, becomes

$$\dot{y}''(x) = \dot{y}_o''(x)\, t \qquad \text{[for very small } t\text{]} \qquad (2)$$

The sign of $\dot{y}''(x)$ is simply the sign of the second derivative of the initial velocity distribution.

Applying this result to the velocity distribution of the example, Fig. 3 of the paper,

$$\dot{y}_o(x) = \frac{v_o}{2}\left(1 + \cos\frac{\pi x}{\ell}\right),$$

it is seen that $\dot{y}''(x)$ is positive for the outer quarters of the beam length, but negative in the center half. The so-called "correct rigid-plastic" analysis indicates therefore that, for early times, angular changes in opposite directions will occur in the various parts of the beam. As t increases and energy is absorbed this behavior will of course be modified in a manner which is the subject of Cotter's paper referred to earlier. For sufficiently large t the beam will be permanently deformed to a shape shown qualitatively in Fig. 1 and will move with a certain final velocity $\dot{y}(x) = v_F$. The center portion will be bent through a total angle θ_1, while each end will be bent in the opposite direction by angles θ_2. (The angle θ^* used by the discusser is not clearly defined; it may be either $\theta^* = \theta_1$, or $\theta^* = \theta_1 - 2\theta_2$. The difference is immaterial in what follows.)

The permanent deformation obtained by the writers is a kink at the center, at the assumed yield hinge. This assumption has been made because the writers expect the permanent deformation of the beams in the examples not to have the character of Fig. 1, but rather of Fig. 2, where all permanent curvatures have the same sign, and occur essentially in the center portion of the beam. If the deformation is of the character of Fig. 2, an assumed yield hinge at the center will clearly give physically meaningful values of the total angle θ for the elasto-plastic case. The value θ_u as defined by the writers will be an upper bound for the elasto-plastic value; the fact that a rigid-plastic approach was used to obtain θ_u does not mean that the result, in general, must be equal to the "rigid-plastic" solution of Miss Cotter. For the velocity distribution of Eq. (3) the results can not possibly agree because considerations of energy require that $\theta_u = \theta_1 + 2\theta_2$ and therefore that $\theta \neq \theta_1$ since $\theta_2 \neq 0$.

The crux of the argument is the writers' contention that the counterflexures obtained in the rigid-plastic approach do not occur in the elasto-plastic beams in either of the examples in the paper. Consider the response of the elastic beam (prior to the plastic stage) and the time history of the bending moment for small values of t. For sufficiently small values of t, the displacements are given again by Eq. (1), and the bending moments become

$$M(x, t) \approx -\, EI\dot{y}_o''(x)t \qquad \text{[for very small } t\text{]} \qquad (4)$$

For the velocity distribution under consideration, Eq. (4) indicates that the largest moments, of opposite signs, occur at $x = 0$ and $x = \pm\ell$. However, the points $x = \pm\ell$ are at the ends of the beam where the moments must necessarily vanish; Eq. (4) therefore does not apply, even for small values of t if $\ell - |x|$ is small. One can, however, visualize the

actual bending moments to be the sum of Eq. (4) and of correcting stress waves originating at $x = \pm \ell$ such that for the sum the condition $M(\pm \ell, t) = 0$ remains satisfied.[2] This suggests that Eq. (4) will apply at points near the ends $x = \pm \ell$ only if t is smaller than the time required for a stress wave to travel the distance $\ell - |x|$. In the present case this stress wave will decrease all moments near the ends, but increase those at the center of the beam. When the time t is larger than ℓ/c, the time required by the waves to travel through the entire beam, Eq. (4) becomes inapplicable for all x, and the bending moments will be redistributed in a manner which need have no similarity to the original curve $\dot{y}_0''(x)$. If first yield takes place at a time $t_0 > \ell$ c, the sign of the bending moments at yield and the direction of the permanent curvature will no longer be governed by the sign of $\dot{y}_0''(x)$. It follows that in the examples considered the type of permanent deformation shown in Fig. 1 can only be obtained if the time to yield is very small compared to ℓ/c. If this is not the case the counterflexing moments will have disappeared by the time yielding begins.

The reasoning in the previous paragraph is physically incomplete because shear deformations have been neglected; for sufficiently small values of the time t this is not permissible. It is not difficult, however to go through the same reasoning for the case of the Timoshenko beam allowing for shear deformations. Excluding cases where yield in shear occurs (which are not contemplated here), one finds again that the yield pattern is governed by the sign of $\dot{y}_0''(x)$ only if the time at yield is small versus ℓ/c, confirming the conclusion found without shear effects.

The so-called "correct rigid-plastic" analysis obtains the location and direction of yielding from the values of $\dot{y}_0''(x)$ and one must conclude that this analysis can not be relied upon when the time to yield in the elasto-plastic structure is of the order of ℓ/c or larger, because in such a case the actual yield pattern is quite different from that obtained by the so-called "correct rigid-plastic" analysis. This difference is important for the examples used in the paper. (It may not be important for other cases, but the question of other cases does not rightly belong here.)

The examples in the paper may be interpreted as applying to steel beams by selecting suitable units, e.g. ft., sec., and kips. The times to first yield, 0.2 sec. and 0.016 sec. respectively are not small versus $\ell/c \approx 0.01$ sec; hence, the so-called "correct rigid-plastic" solution is not applicable in either case. It should be clear that the time to yield would have to be considerably smaller than ℓ/c to make the rigid-plastic solution applicable.

The reviewer asks also why the writers have not discussed "methods for examining all cross-sections of the beam." No special "methods" are required. Once the coordinates q_i or p_i are obtained the moment history can be plotted by routine computation, if one desires to do so. Some points of such curves were computed, but not included in the paper as they did not reveal anything of particular interest. The raising of the above question, and the discusser's emphasis on checking the capacity moments of adjoining sections shows a lack of realization of the type of

2. To be consistent, the beam should be deemed to have rotatory inertia and to be infinitely rigid in shear. Eq. (4) does not visualize shear deformations.

problem which should and can be treated by normal mode methods in general, with or without plasticity. If one desires to treat a problem in which information at many points is required with good accuracy, normal mode methods are impractical even in the elastic case. On the other hand, if one is looking for approximate, yet physically reasonable results, mode methods come into their own, because one can restrict the computations to a small number of modes. Such a restriction results in a loss of detail and in problems of elasto-plastic beams it would be absurd to pay attention to minor motion of hinges or areas becoming plastic; any conclusions as to such matters on the basis of a few terms of an expression would be quite unrealistic. It is only consistent with the limited accuracy obtainable to place a yield hinge in a suitably selected strategic position, or in several positions if necessary; one will still get useful and meaningful results. This is the reason for the selection of a yield hinge at the center only in the examples treated; it represents the essential physical effects. If the initial velocity in the examples were increased considerably it might of course be necessary to consider two additional hinges in the outer quarters of the beam to allow for permanent deformations according to Fig. 1. Cotter's rigid-plastic solution is an excellent basis for the selection of the best location of these hinges. For sufficiently large v_0 where the rigid-plastic solution is applicable there is of course no point in even trying to use a mode solution.

To sum up, it is the writers' position that both elasto-plastic and rigid-plastic methods have their place. The elasto-plastic approach is necessary in the very important practical cases where the loads exceed the elastic capacity by a modest amount only, since a rigid-plastic analysis (which is necessarily simpler) is inapplicable to such cases. Moreover, the writers point out that in cases of initial velocities an additional criterion may be required for the rigid-plastic solution to be valid. (In the original publication by Lee and Symonds the energy criterion was not really proposed for cases of initial velocity, since these cases were not treated there at all.)

In the very last paragraph of the discussion the discusser raises the question of simple beams on which subject he is preparing a paper. It seems best to postpone the consideration of this question until this paper is available.

Postscript

After submission of the preceding closure, an advance copy of Miss Cotter's paper to which the discusser refers was made available to the writers. One of the results of this paper, not mentioned by the discusser, is of considerable interest to this closure and is therefore added here.

Miss Cotter found that the deformation of the bar is of the type shown in Fig. 1 but that the total permanent deformation θ_1 near the center of the bar consists of two parts: one part is a kink at the center, the other a bend distributed over a length of about $\ell/6$.

In comparing Miss Cotter's results and those of the writers, the discusser has quoted only the value of the kink-angle at the center; this

angle differs from the kink-angle found by the writers on the basis of the single hinge analysis by the stated amounts. Miss Cotter's paper also contains the values for the distributed bending in the vicinity of the center, and the total angles found by Miss Cotter and by the writers differ only very slightly:

Initial velocity v_0 =	11.1	67.4
Miss Cotter:	$2.64\ v_0\ 10^{-3}$	$1.60\ v_0\ 10^{-2}$
Bleich and Salvadori:	$2.67\ v_0\ 10^{-3}$	$1.61\ v_0\ 10^{-2}$

The "so-called correct rigid-plastic" analysis differs therefore from the writers' upper limit solution essentially only by the fact that some of the deformation is distributed over a short distance rather than concentrated as a kink. It is worth recalling that the occurrence of any kink is only due to the simplification in the moment-curvature relationship, and that for purposes of application to any actual structure only the total angle is physically meaningful. This indicates that the differences between the two rigid-plastic analyses are pretty close to trivial.

The above comparison is not intended to withdraw the writers' argument in the main body of this closure, stressing that the "so-called correct rigid-plastic" analysis is not applicable in either of the examples because the additional criterion stated is not satisified.

In conclusion, the writers believe to have shown that the discusser's criticism of the results of their paper is not justified.

CORRECTIONS for Transactions.—On the front cover page, the initials of M. G. Salvadori have been printed incorrectly. In Eq. 13, the double-prime superscript should be outside the closing parenthesis. In Fig. 8, the expression at the top of the ordinate scale should be $-\frac{10^2\theta_0}{v_0}$.

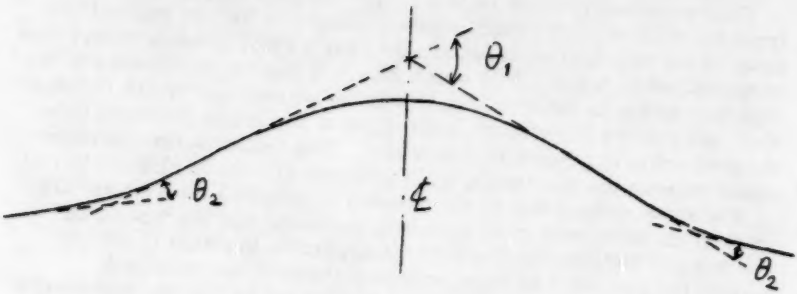

Fig. 1

Fig. 2

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a. Presented at the New York (N.Y.) Convention of the Society in October, 1953.
b. Beginning with "Proceedings-Separate No. 290," published in October, 1953, an automatic distribution of papers was inaugurated, as outlined in "Civil Engineering," June, 1953, page 66.
c. Discussion of several papers, grouped by Divisions.
d. Presented at the Atlanta (Ga.) Convention of the Society in February, 1954.
e. Presented at the Atlantic City (N.J.) Convention in June, 1954.

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